REPLICATOR-MUTATOR EQUATIONS WITH QUADRATIC FITNESS

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Abstract. This work completes our previous analysis on models arising in evolutionary genetics. We consider the so-called replicator-mutator equation, when the fitness is quadratic. This equation is a heat equation with a harmonic potential, plus a specific nonlocal term. We give an explicit formula for the solution, thanks to which we prove that when the fitness is non-positive (harmonic potential), solutions converge to a universal stationary Gaussian for large time, whereas when the fitness is non-negative (inverted harmonic potential), solutions always become extinct in finite time.

1. Introduction

This note is concerned with *replicator-mutator* equations, that is nonlocal reaction diffusion problems of the form

$$\partial_t U = {\sigma_0}^2 \partial_{xx} U + \mu_0 \left(f(x) - \int_{\mathbb{R}} f(x) U(t,x)\, dx \right) U, \quad t>0,\ x \in \mathbb{R},$$

where $\sigma_0 > 0$ and $\mu_0 > 0$ are parameters, and when either $f(x) = -x^2$ or $f(x) = x^2$. In order to simplify the presentation of the results, and before going into more details, we use the rescaling

$$u(t,x) := U\left(\frac{t}{\mu_0}, x \right), \quad \sigma := \frac{\sigma_0}{\sqrt{\mu}_0},$$

and therefore consider

$$\partial_t u = \sigma^2 \partial_{xx} u + (f(x) - \overline{f}(t))u, \quad t>0,\ x\in\mathbb{R}, \tag{1.1}$$

where the nonlocal term is given by

$$\overline{f}(t) := \int_{\mathbb{R}} f(x) u(t,x)\, dx. \tag{1.2}$$

Equation (1.1) is always supplemented with an initial condition $u_0 \geq 0$ with mass $\int_{\mathbb{R}} u_0 = 1$, so that the mass is *formally* conserved for later times. Indeed, integrating (1.1) over $x \in \mathbb{R}$, we find that $m(t) := \int_{\mathbb{R}} u(t,x)dx$ satisfies

$$\frac{dm}{dt} = \overline{f}(t) \left(1 - m(t)\right), \quad m(0) = 1,$$

hence $m(t) = 1$ so long as $\overline{f}$ is integrable.

In the context of evolutionary genetics, Equation (1.1) was introduced by Tsimring et al. [17], where they propose a mean-field theory for the evolution of RNA

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virus populations on a phenotypic trait space. In this context, $u(t,x)$ is the density of a population (at time t and per unit of phenotypic trait) on a one-dimensional phenotypic trait space, and $f(x)$ represents the fitness of an individual with trait value x in a population which is at state $u(t,x)$. The nonlocal term $\overline{f}(t)$ represents the mean fitness at time t. We refer to [1] for more references on the biological background of (1.1).

1.1. **The case** $f(x)=x$**.** This case can be seen as a parabolic counterpart of the Schrödinger equation with a Stark potential; in the context of quantum mechanics, this potential corresponds to a constant electric field or to gravity (see e.g. [16]). In the case of (1.1), a family of self similar Gaussian solutions has been constructed in [2]. Then this case has been completely studied in [1]. It turns out that not only traveling pulses are changing sign, but also extinction in finite time occurs for initial data with "not very light tails" (data which do not decay very fast on the right). This, in particular, contradicts the formal conservation of the mass observed in (1.1) and evoked above. Roughly speaking, the nonlocal term of the equation $\int_{\mathbb{R}} xu(t,x)dx$ becomes infinite and the equation becomes meaningless. On the other hand, for initial data with "very light tails" (they decay sufficiently fast on the right), the solution is defined for all times $t \geq 0$ and is escaping to the right by accelerating and flattening as $t \to \infty$. More precisely, the long time behaviour is (when $\sigma = 1$) a Gaussian centered at $x(t) = t^2$ (acceleration) and of maximal height $1/\sqrt{4\pi t}$ (flattening effect). In other words, extinction occurs at $t = \infty$ in this situation. This is like in the case of the linear heat equation, up to the fact that the center of the asymptotic Gaussian is given by $x(t)$, which undergoes some acceleration which is reminiscent of the effect of gravity. Notice some links of this acceleration phenomena with some aspects of the so-called *dynamics of the fittest trait* (see [6], [8], [12] and the references therein) which, in some cases, escape to infinity for large times [7], [15].

1.2. **The quadratic cases.** To prove the above results in [1], we used a change of unknown function based on the Avron–Herbst formula for the Schrödinger equation, and showed that (1.1) is equivalent to the heat equation. We could then compute its solution explicitly. Without those exact computations, the understanding of (1.1) seems far from obvious, and in particular the role of the decay on the right of the initial data. In [1] we indicated that similar computations could also be performed in the cases $f(x) = \pm x^2$, thanks to the generalized lens transform of the Schrödinger equation, but without giving any detail. The goal of the present work is to fill this gap, by giving the full details and results for these two cases.

Our motivation is twofold. First, very recently, replicator-mutator equations (or related problems) with quadratic fitness have attracted a lot of attention: let us mention the works [10], [3], [5], [11], [9], [18] and the references therein. In particular, Chisholm et al. [5] study — among other things — the long time behaviour of the nonlocal term of an equation very close to (1.1), in the case $f(x) = -x^2$, with compactly supported initial data. In Section 2 we completely solve (1.1) for any initial data, and can therefore study the long time behaviour not only of the nonlocal term $\overline{f}(t)$ but also of the full profile $u(t,x)$. The second reason is that the obtained behaviours are varied and interesting, bringing precious information in the dynamical study of partial differential equations. To give a preview of this, we now state two theorems which are direct consequences of the more detailed results of Section 2 ($f(x) = -x^2$) and Section 3 ($f(x) = x^2$).

In the case $f(x) = -x^2$, solutions tend at large time to a universal stationary Gaussian. Denote

$$\mathcal{M}_2(\mathbb{R}) := \left\{ g \in L^1(\mathbb{R}), \quad \int_{\mathbb{R}} x^2 |g(x)| dx < \infty \right\}.$$

Theorem 1.1 (Case $f(x) = -x^2$)**.** *Let $u_0 \geq 0$, with $\int_{\mathbb{R}} u_0 = 1$. Then* (1.1)*, with initial datum u_0, has a unique solution $u \in C(\mathbb{R}_+; L^1(\mathbb{R})) \cap L^1_{\rm loc}((0,\infty); \mathcal{M}_2(\mathbb{R}))$. It satisfies*

$$\sup_{x \in \mathbb{R}} |u(t,x) - \varphi(x)| \underset{t \to \infty}{\longrightarrow} 0, \quad \textit{where } \varphi(x) := \frac{1}{\sqrt{2\pi\sigma}} e^{-x^2/(2\sigma)}.$$

The above result shows that the presence of the quadratic fitness compensates the diffusive mechanism of the heat equation, since we recall that for $u_0 \geq 0$ with $\int (1+|x|) u_0(x) dx < \infty$,

$$e^{t\partial_{xx}} u_0(x) = \frac{\|u_0\|_{L^1(\mathbb{R})}}{\sqrt{4\pi t}} e^{-x^2/(4t)} + o(1), \quad \text{in } L^\infty(\mathbb{R}), \text{ as } t \to \infty.$$

We also emphasize that in Theorem 1.1, we do not assume that u_0 has two momenta in $L^1(\mathbb{R})$, $u_0 \in \mathcal{M}_2(\mathbb{R})$: this property is satisfied by the solution instantaneously, as we will see in Section 2.

On the other hand, in the case $f(x) = x^2$, extinction in finite time always occurs.

Theorem 1.2 (Case $f(x) = x^2$)**.** *Let $u_0 \geq 0$, with $\int_{\mathbb{R}} u_0 = 1$. Then the solution to* (1.1)*, with initial datum u_0, becomes extinct in finite time:*

$$\exists T \in \left[0, \frac{\pi}{4\sigma}\right], \quad u(t,x) = 0, \quad \forall t > T, \ \forall x \in \mathbb{R}.$$

As in [1], the solution may become extinct *instantaneously* ($T = 0$), that is, (1.1) has no non-trivial solution, if the initial datum has too little decay at infinity.

1.3. **Heat vs. Schrödinger.** As mentioned above, the present results, as well as those established in [1], stem from explicit formulas discovered in the context of Schrödinger equations; see [13, 14], [16]. However, we have to emphasize several differences between (1.1) and its Schrödinger analogue,

$$i\partial_t u = \sigma^2 \partial_{xx} u + (f(x) - \overline{f}(t))u, \quad t \in \mathbb{R}, \ x \in \mathbb{R}. \tag{1.3}$$

The Schrödinger equation is of course time reversible. Less obvious is the way the term $\overline{f}(t)u$ is handled, according to the equation one considers. In the Schrödinger case (1.3), we simply use a gauge transform to get rid of this term: it is equivalent to consider u solution to (1.3) or

$$v(t,x) = u(t,x) e^{-i\int_0^t \overline{f}(s)ds}, \tag{1.4}$$

which solves

$$i\partial_t v = \sigma^2 \partial_{xx} v + f(x) v,$$

with the same initial datum. If we assume, like in the case of (1.1), that $\overline{f}(t)$ is real (which means that it is *not* given by (1.2) in this case), the change of unknown function (1.4) does not alter the dynamics, since $|v(t,x)| = |u(t,x)|$. On the other hand, the analogous transformation in the parabolic case has a true effect on the dynamics since, as pointed out in [1], it becomes

$$v(t,x) = u(t,x)e^{\int_0^t \overline{f}(s)ds}. \tag{1.5}$$

Multiplying by $f(x)$ and integrating in x, we infer

$$\int_{\mathbb{R}} f(x)v(t,x)dx = \overline{f}(t)e^{\int_0^t \overline{f}(s)ds} = \frac{d}{dt}\left(e^{\int_0^t \overline{f}(s)ds}\right).$$

Therefore, *so long as* $\int_0^t \int_{\mathbb{R}} f(x)v(s,x)dxds > -1$,

$$u(t,x) = \frac{v(t,x)}{1+\int_0^t \int_{\mathbb{R}} f(x)v(s,x)dxds}.$$

It is clear that in general, u and v now have different large time behaviours.

The last algebraic step to construct explicit solutions for (1.1) and (1.3) consists in using the Avron–Herbst formula when $f(x) = x$, or a (generalized) lens transform when $f(x) = \pm x^2$. In the case of the standard (quantum) harmonic oscillator, the solutions to

$$i\partial_t v + \partial_{xx} v = \omega^2 x^2 v, \quad \text{and} \quad i\partial_t w + \partial_{xx} w = 0, \quad \text{with } v_{|t=0} = w_{|t=0},$$

are related through the formula

$$v(t,x) = \frac{1}{\sqrt{\cos(2\omega t)}} w\left(\frac{\tan(2\omega t)}{2\omega}, \frac{x}{\cos(2\omega t)}\right) e^{-i\frac{\omega}{2}x^2\tan(2\omega t)}, \quad |t| < \frac{\pi}{4\omega}.$$

What this formula does not show, since it is limited in time, is that the solution v is periodic in time, as can be seen for instance by considering an eigenbasis for the harmonic oscillator $-\partial_{xx} + \omega^2 x^2$, given by Hermite functions. Suppose $\omega = 1$ to lighten the notations: the Hermite functions $(\psi_j)_{j\geq 0}$ form an orthogonal basis of $L^2(\mathbb{R})$, and

$$-\partial_{xx}\psi_j + x^2\psi_j = (2j+1)\psi_j.$$

Therefore, if

$$v(0,x) = \sum_{j\geq 0} \alpha_j \psi_j(x), \quad \text{then } v(t,x) = \sum_{j\geq 0} \alpha_j \psi_j(x) e^{i(2j+1)t}$$

is obviously 2π-periodic in time. This is in sharp contrast with the behaviour described in Theorem 1.1. Similarly, the solutions to

$$i\partial_t v + \partial_{xx} v = -\omega^2 x^2 v, \quad i\partial_t w + \partial_{xx} w = 0, \quad v_{|t=0} = w_{|t=0},$$

are related through the formula (change ω to $i\omega$ in the previous formula)

$$v(t,x) = \frac{1}{\sqrt{\cosh(2\omega t)}} w\left(\frac{\tanh(2\omega t)}{2\omega}, \frac{x}{\cosh(2\omega t)}\right) e^{i\frac{\omega}{2}x^2\tanh(2\omega t)}, \quad t \in \mathbb{R}.$$

This shows that the inverted harmonic potential accelerates the dispersion ($\|v(t,\cdot)\|_{L^\infty}$ goes to zero exponentially fast), and the large time profile is given by $w_{|t=1/(2\omega)}$. Again, this behaviour is completely different from the one stated in Theorem 1.2.

2. The case $f(x) = -x^2$: convergence to a universal Gaussian

The case $f(x) = -x^2$ can be handled as explained in [1]. We give more details here, and analyze the consequences of the explicit formula. In particular, for any initial data, the solution is defined for all positive times and converge, at large time, to a universal stationary Gaussian.

2.1. **Results.**

Theorem 2.1 (The solution explicitly). *Let $u_0 \geq 0$, with $\int_{\mathbb{R}} u_0 = 1$. Then* (1.1) *with initial datum u_0 has a unique solution $u \in C(\mathbb{R}_+; L^1(\mathbb{R})) \cap L^1_{\rm loc}((0,\infty); \mathcal{M}_2(\mathbb{R}))$. For all $t > 0$ and $x \in \mathbb{R}$, it is given by*

$$u(t,x) = \frac{1}{\sqrt{2\pi\sigma \tanh(2\sigma t)}} \frac{e^{-\frac{\tanh(2\sigma t)}{2\sigma}x^2} \displaystyle\int_{\mathbb{R}} e^{-\frac{1}{2\sigma \tanh(2\sigma t)}\left(\frac{x}{\cosh(2\sigma t)} - y\right)^2} u_0(y)\, dy}{\displaystyle\int_{\mathbb{R}} e^{-\frac{\tanh(2\sigma t)}{2\sigma}y^2} u_0(y)\, dy} \tag{2.1}$$

$$= \frac{1}{\sqrt{2\pi\sigma \tanh(2\sigma t)}} \frac{\displaystyle\int_{\mathbb{R}} e^{-\frac{1}{2\sigma \tanh(2\sigma t)}\left(x - \frac{y}{\cosh(2\sigma t)}\right)^2} e^{-\frac{\tanh(2\sigma t)}{2\sigma}y^2} u_0(y)\, dy}{\displaystyle\int_{\mathbb{R}} e^{-\frac{\tanh(2\sigma t)}{2\sigma}y^2} u_0(y)\, dy}. \tag{2.2}$$

As a consequence, for all $t > 0$, $\overline{f}(t)$ is given by

$$\overline{f}(t) = \sigma \tanh(2\sigma t) + \frac{1}{(\cosh(2\sigma t))^2} \frac{\displaystyle\int_{\mathbb{R}} e^{-\frac{\tanh(2\sigma t)}{2\sigma}y^2} y^2\, u_0(y)\, dy}{\displaystyle\int_{\mathbb{R}} e^{-\frac{\tanh(2\sigma t)}{2\sigma}y^2} u_0(y)\, dy}. \tag{2.3}$$

We now investigate the propagation of Gaussian initial data.

Proposition 2.2 (Propagation of Gaussian initial data). *If*

$$u_0(x) = \sqrt{\frac{a}{2\pi}} e^{-\frac{a}{2}(x-m)^2}, \quad a > 0, \quad m \in \mathbb{R}, \tag{2.4}$$

then the solution of (1.1) *remains Gaussian for $t > 0$ and is given by*

$$u(t,x) = \sqrt{\frac{a(t)}{2\pi}} e^{-\frac{a(t)}{2}(x-m(t))^2}, \tag{2.5}$$

where

$$a(t) := \frac{a\sigma + \tanh(2\sigma t)}{\sigma(1 + a\sigma \tanh(2\sigma t))}, \quad m(t) := \frac{ma\sigma}{a\sigma \cosh(2\sigma t) + \sinh(2\sigma t)}. \tag{2.6}$$

Since $a(t) \to \frac{1}{\sigma}$ and $m(t) \to 0$ as $t \to \infty$, it is easily seen that $u(t,x) \to \varphi(x) := \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2\sigma}x^2}$ uniformly in $x \in \mathbb{R}$. This fact is actually true for *all* initial data, as stated in the following theorem, which implies Theorem 1.1.

Theorem 2.3 (Long time behaviour). *Under the assumptions of Theorem 2.1, there exists $C > 0$ independent of time such that*

$$\sup_{x \in \mathbb{R}} |u(t,x) - \psi(t,x)| \leq \frac{C}{\sinh(2\sigma t)}, \quad \forall t \geq 1, \tag{2.7}$$

where

$$\psi(t,x) := \frac{1}{\sqrt{2\pi\sigma \tanh(2\sigma t)}} e^{-\frac{1}{2\sigma \tanh(2\sigma t)}x^2}$$

satisfies

$$\psi(t,x) \underset{t \to \infty}{\longrightarrow} \varphi(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2\sigma}x^2},$$

uniformly in $x \in \mathbb{R}$.

Observe that $\psi(t,x)$ is nothing but the fundamental solution obtained by plugging $u_0(y) = \delta_0(y)$, the Dirac mass at the origin, in (2.2), so the first part of the statement is the analogue of the convergence result recalled in the introduction,

$$\left\| e^{t\partial_{xx}} u_0(x) - \frac{\|u_0\|_{L^1(\mathbb{R})}}{\sqrt{4\pi t}} e^{-x^2/(4t)} \right\|_{L^\infty(\mathbb{R})} \leq \frac{C}{t} \|xu_0\|_{L^1(\mathbb{R})},$$

and the effect of the fitness $f(x) = -x^2$ is to neutralize diffusive effects.

2.2. **Proofs.**

Proof of Theorem 2.1. As proved in [1], we can reduce (1.1) to the heat equation by combining two changes of unknown function. First, we have

$$u(t,x) = \frac{v(t,x)}{1 - \displaystyle\int_0^t \int_{\mathbb{R}} x^2 v(s,x)\, dxds}, \tag{2.8}$$

where $v(t,x)$ solves the Cauchy problem

$$\partial_t v = \sigma^2 \partial_{xx} v - x^2 v, \quad t > 0,\ x \in \mathbb{R}; \quad v(0,x) = u_0(x).$$

Notice that relation (2.8) is valid as long as $\int_0^t \int_{\mathbb{R}} x^2 v(s,x)\, dxds < 1$. Next, by adapting the so-called lens transform ([13], [4]), we have

$$v(t,x) = \frac{1}{\sqrt{\cosh(2\sigma t)}} e^{-\frac{\tanh(2\sigma t)}{2\sigma}x^2} w\left(\frac{\tanh(2\sigma t)}{2\sigma}, \frac{x}{\sigma\cosh(2\sigma t)}\right), \tag{2.9}$$

where $w(t,x)$ solves the heat equation

$$\partial_t w = \partial_{xx} w, \quad t > 0,\ x \in \mathbb{R}; \quad w(0,x) = u_0(\sigma x).$$

Combining (2.8), (2.9) and the integral expression of w via the heat kernel, we end up with

$$\begin{aligned} &u(t,x) = \frac{1}{1 - I(t)} \times \\ &\sqrt{\frac{\sigma}{2\pi}} \frac{1}{\sqrt{\sinh(2\sigma t)}} e^{-\frac{\tanh(2\sigma t)}{2\sigma}x^2} \int_{\mathbb{R}} e^{-\frac{\sigma}{2\tanh(2\sigma t)}\left(\frac{x}{\sigma\cosh(2\sigma t)} - y\right)^2} u_0(\sigma y)\, dy, \end{aligned} \tag{2.10}$$

where

$$I(t) := \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} x^2 \sqrt{\frac{\sigma}{2\pi}} \frac{1}{\sqrt{\sinh(2\sigma s)}} e^{-\frac{\tanh(2\sigma s)}{2\sigma}x^2} e^{-\frac{\sigma}{2\tanh(2\sigma s)}\left(\frac{x}{\sigma\cosh(2\sigma s)} - y\right)^2} u_0(\sigma y)\, dydxds.$$

Let us compute $I(t)$. Using Fubini's theorem, we first compute the integral with respect to x. Using elementary algebra (canonical form in particular), we get

$$\begin{aligned} &\int_{\mathbb{R}} x^2 e^{-\frac{\tanh(2\sigma s)}{2\sigma}x^2} e^{-\frac{\sigma}{2\tanh(2\sigma s)}\left(\frac{x}{\sigma\cosh(2\sigma s)} - y\right)^2} dx \\ &= e^{-\frac{\sigma\tanh(2\sigma s)}{2}y^2} \int_{\mathbb{R}} x^2 e^{-\frac{1}{2\sigma\tanh(2\sigma s)}\left(x - \frac{\sigma y}{\cosh(2\sigma s)}\right)^2} dx \\ &= e^{-\frac{\sigma\tanh(2\sigma s)}{2}y^2} \sqrt{2\pi\sigma\tanh(2\sigma s)} \left(\sigma\tanh(2\sigma s) + \frac{\sigma^2 y^2}{(\cosh(2\sigma s))^2}\right), \end{aligned} \tag{2.11}$$

where we have used the straightforward formula $\int_{\mathbb{R}} z^2 e^{-\frac{a}{2}(z-\theta)^2} dz = \frac{1}{a}\sqrt{\frac{2\pi}{a}} + \theta^2\sqrt{\frac{2\pi}{a}}$. Next, we pursue the computation of $I(t)$ and, integrating with respect to s, we find

$$\int_0^t \sqrt{\frac{\sigma}{2\pi}} \frac{1}{\sqrt{\sinh(2\sigma s)}} e^{-\frac{\sigma \tanh(2\sigma s)}{2} y^2} \sqrt{2\pi\sigma\tanh(2\sigma s)} \left(\sigma\tanh(2\sigma s) + \frac{\sigma^2 y^2}{(\cosh(2\sigma s))^2}\right) ds$$
$$= \sigma \int_0^t e^{-\frac{1}{2}\ln(\cosh(2\sigma s))} e^{-\frac{\sigma\tanh(2\sigma s)}{2} y^2} \left(\sigma\tanh(2\sigma s) + \frac{\sigma^2 y^2}{(\cosh(2\sigma s))^2}\right) ds$$
$$= \sigma \int_0^t \frac{d}{ds}\left(-e^{-\frac{1}{2}\ln(\cosh(2\sigma s))} e^{-\frac{\sigma\tanh(2\sigma s)}{2} y^2}\right) ds$$
$$= \sigma\left(1 - \frac{1}{\sqrt{\cosh(2\sigma t)}} e^{-\frac{\sigma\tanh(2\sigma t)}{2} y^2}\right).$$

Finally, we integrate with respect to y and, using $\int_{\mathbb{R}} u_0 = 1$, get

$$I(t) = \int_{\mathbb{R}} \sigma\left(1 - \frac{1}{\sqrt{\cosh(2\sigma t)}} e^{-\frac{\sigma\tanh(2\sigma t)}{2} y^2}\right) u_0(\sigma y)\, dy$$
$$= 1 - \frac{1}{\sqrt{\cosh(2\sigma t)}} \int_{\mathbb{R}} e^{-\frac{\tanh(2\sigma t)}{2\sigma} z^2} u_0(z)\, dz.$$

Plugging this in the denominator of (2.10), and using the change of variable $z = \sigma y$ in the numerator of (2.10), we get (2.1), from which (2.2) easily follows. Using (2.2), Fubini theorem and the same computation as in (2.11), we obtain (2.3). The solution thus obtained satisfies $u \in C(\mathbb{R}_+; L^1(\mathbb{R})) \cap L^1_{\text{loc}}((0,\infty); \mathcal{M}_2(\mathbb{R}))$. Uniqueness for such a solution stems from the transformations that we have used, which require exactly this regularity (see also (1.5)). Theorem 2.1 is proved. □

Proof of Proposition 2.2. We plug the Gaussian data (2.4) into formula (2.1). Using elementary algebra (canonical form), we first compute

$$\int_{\mathbb{R}} e^{-\frac{\tanh(2\sigma t)}{2\sigma} y^2} u_0(y)\, dy$$
$$= \sqrt{\frac{a}{2\pi}} e^{-\frac{m^2}{2}\left(a - \frac{a^2}{\frac{\tanh(2\sigma t)}{\sigma} + a}\right)} \int_{\mathbb{R}} e^{-\frac{1}{2}\left(\frac{\tanh(2\sigma t)}{\sigma} + a\right)\left(y - \frac{am}{\frac{\tanh(2\sigma t)}{\sigma} + a}\right)^2} dy$$
$$= \sqrt{\frac{a\sigma}{\tanh(2\sigma t) + a\sigma}} e^{-\frac{m^2}{2}\frac{a\tanh(2\sigma t)}{\tanh(2\sigma t) + a\sigma}}.$$

Some tedious but similar manipulations involving canonical form imply

$$\int_{\mathbb{R}} e^{-\frac{1}{2\sigma\tanh(2\sigma t)}\left(\frac{x}{\cosh(2\sigma t)} - y\right)^2} u_0(y)\, dy$$
$$= \sqrt{\frac{a}{2\pi}} e^{-\frac{a}{(\cosh(2\sigma t))^2(1 + a\sigma\tanh(2\sigma t))}\frac{x^2}{2}} e^{\frac{am}{\cosh(2\sigma t)(1 + a\sigma\tanh(2\sigma t))} x} e^{-\frac{a}{2(1 + a\sigma\tanh(2\sigma t))} m^2}$$
$$\times \int_{\mathbb{R}} e^{-\frac{1 + a\sigma\tanh(2\sigma t)}{2\sigma\tanh(2\sigma t)}\left[y - \frac{\sigma\tanh(2\sigma t)}{1 + a\sigma\tanh(2\sigma t)}\left(\frac{x}{\sigma\tanh(2\sigma t)\cosh(2\sigma t)} + am\right)\right]^2} dy$$
$$= \sqrt{\frac{a\sigma\tanh(2\sigma t)}{1 + a\sigma\tanh(2\sigma t)}} e^{-\frac{a}{(\cosh(2\sigma t))^2(1 + a\sigma\tanh(2\sigma t))}\frac{x^2}{2}} e^{\frac{am}{\cosh(2\sigma t)(1 + a\sigma\tanh(2\sigma t))} x} e^{-\frac{a}{2(1 + a\sigma\tanh(2\sigma t))} m^2}.$$

Putting all together into (2.1), we arrive, after computations involving hyperbolic functions, at the desired formulas (2.5) and (2.6). □

Proof of Theorem 2.3. Since $\psi(t,x)$ is nothing but the fundamental solution arising from (2.2) with $u_0(y)=\delta_0(y)$, we write

$$\psi(t,x)=\frac{1}{\sqrt{2\pi\sigma\tanh(2\sigma t)}}\frac{\displaystyle\int_{\mathbb{R}} e^{-\frac{1}{2\sigma\tanh(2\sigma t)}x^2}e^{-\frac{\tanh(2\sigma t)}{2\sigma}y^2}u_0(y)\,dy}{\displaystyle\int_{\mathbb{R}} e^{-\frac{\tanh(2\sigma t)}{2\sigma}y^2}u_0(y)\,dy},$$

so that the deviation from this fundamental solution is given by

$$\begin{aligned}&(u(t,x)-\psi(t,x))\sqrt{2\pi\sigma\tanh(2\sigma t)}\\ &=\frac{\displaystyle\int_{\mathbb{R}}\left(e^{-\frac{1}{2\sigma\tanh(2\sigma t)}\left(x-\frac{y}{\cosh(2\sigma t)}\right)^2}-e^{-\frac{1}{2\sigma\tanh(2\sigma t)}x^2}\right)e^{-\frac{\tanh(2\sigma t)}{2\sigma}y^2}u_0(y)\,dy}{\displaystyle\int_{\mathbb{R}} e^{-\frac{\tanh(2\sigma t)}{2\sigma}y^2}u_0(y)\,dy}.\end{aligned}\tag{2.12}$$

Define $G(z):=e^{-z^2/(2\sigma)}$. It follows from the mean value theorem that

$$|u(t,x)-\psi(t,x)|\sqrt{2\pi\sigma\tanh(2\sigma t)}\le\|G'\|_\infty\frac{\displaystyle\int_{\mathbb{R}}\frac{|y|}{\cosh(2\sigma t)\sqrt{\tanh(2\sigma t)}}e^{-\frac{\tanh(2\sigma t)}{2\sigma}y^2}u_0(y)\,dy}{\displaystyle\int_{\mathbb{R}} e^{-\frac{\tanh(2\sigma t)}{2\sigma}y^2}u_0(y)\,dy},$$

which in turn implies

$$\begin{aligned}|u(t,x)-\psi(t,x)|&\le\frac{\|G'\|_\infty}{\sqrt{2\pi\sigma}\sinh(2\sigma t)}\frac{\displaystyle\int_{\mathbb{R}} e^{-\frac{\tanh(2\sigma t)}{2\sigma}y^2}|y|u_0(y)\,dy}{\displaystyle\int_{\mathbb{R}} e^{-\frac{\tanh(2\sigma t)}{2\sigma}y^2}u_0(y)\,dy}\\ &\le\frac{\|G'\|_\infty}{\sqrt{2\pi\sigma}\sinh(2\sigma t)}\frac{\displaystyle\int_{\mathbb{R}} e^{-\frac{\tanh(2\sigma)}{2\sigma}y^2}|y|u_0(y)\,dy}{\displaystyle\int_{\mathbb{R}} e^{-\frac{1}{2\sigma}y^2}u_0(y)\,dy}=:\frac{C}{\sinh(2\sigma t)},\end{aligned}$$

for all $t\ge 1$. This proves (2.7). □

3. The case $f(x)=x^2$: systematic extinction in finite time

The case $f(x)=x^2$ can be handled as explained in [1]. Details are presented below: for any initial datum, the solution becomes extinct in finite time.

Indeed, it will turn out that there are two limitations for the time interval of existence of the solution. The first limitation arises when reducing equation (3.9) to the heat equation (3.11) through the relation (3.10), which requires

$$0<t<T^{\text{Heat}}:=\frac{\pi}{4\sigma}.$$

The other limitation appears when reducing (1.1) to (3.9), which requires

$$\int_{\mathbb{R}} e^{\frac{\tan(2\sigma t)}{2\sigma}y^2}u_0(y)dy$$

to remain finite (otherwise the solution becomes extinct). Hence, for $u_0 \geq 0$ with $\int_{\mathbb{R}} u_0 = 1$, we define

$$T := \sup\left\{0 \leq t < T^{\text{Heat}}, \quad \int_{\mathbb{R}} e^{\frac{\tan(2\sigma t)}{2\sigma} y^2} u_0(y) dy < \infty\right\} \in [0, T^{\text{Heat}}]. \tag{3.1}$$

Some typical situations are the following: if $u_0(x)$ has algebraic or exponential tails then $T = 0$ (immediate extinction); if $u_0(x)$ has Gaussian tails then $0 < T < T^{\text{Heat}}$ (rapid extinction in finite time); last, if $u_0(x)$ is compactly supported or has "very very light tails" then $T = T^{\text{Heat}}$ (extinction in finite time).

3.1. **Results.**

Theorem 3.1 (The solution explicitly)**.** *Let $u_0 \geq 0$, with $\int_{\mathbb{R}} u_0 = 1$. As long as $\overline{f}(t)$ is finite, the solution of* (1.1) *with initial data u_0 is given by*

$$u(t,x) = \frac{1}{\sqrt{2\pi\sigma\tan(2\sigma t)}} \frac{e^{\frac{\tan(2\sigma t)}{2\sigma} x^2} \displaystyle\int_{\mathbb{R}} e^{-\frac{1}{2\sigma\tan(2\sigma t)}\left(\frac{x}{\cos(2\sigma t)} - y\right)^2} u_0(y)\, dy}{\displaystyle\int_{\mathbb{R}} e^{\frac{\tan(2\sigma t)}{2\sigma} y^2} u_0(y)\, dy} \tag{3.2}$$

$$= \frac{1}{\sqrt{2\pi\sigma\tan(2\sigma t)}} \frac{\displaystyle\int_{\mathbb{R}} e^{-\frac{1}{2\sigma\tan(2\sigma t)}\left(x - \frac{y}{\cos(2\sigma t)}\right)^2} e^{\frac{\tan(2\sigma t)}{2\sigma} y^2} u_0(y)\, dy}{\displaystyle\int_{\mathbb{R}} e^{\frac{\tan(2\sigma t)}{2\sigma} y^2} u_0(y)\, dy}. \tag{3.3}$$

As long as it exists, $\overline{f}(t)$ is given by

$$\overline{f}(t) = \sigma\tan(2\sigma t) + \frac{1}{(\cos(2\sigma t))^2} \frac{\displaystyle\int_{\mathbb{R}} e^{\frac{\tan(2\sigma t)}{2\sigma} y^2} y^2\, u_0(y)\, dy}{\displaystyle\int_{\mathbb{R}} e^{\frac{\tan(2\sigma t)}{2\sigma} y^2} u_0(y)\, dy}. \tag{3.4}$$

Remark 3.2. Formally, one can notice that $-x^2$ is turned into $+x^2$ in (1.1) if one changes σ to $i\sigma$, and t to $-t$. After such transforms, (2.1)-(2.2) becomes (3.2)-(3.3).

Proposition 3.3 (Propagation of Gaussian initial data)**.** *If*

$$u_0(x) = \sqrt{\frac{a}{2\pi}} e^{-\frac{a}{2}(x-m)^2}, \quad a > 0, \quad m \in \mathbb{R}, \tag{3.5}$$

then the solution of (1.1) *remains Gaussian for*

$$0 < t < T = \frac{\arctan(a\sigma)}{2\sigma},$$

and is given by

$$u(t,x) = \sqrt{\frac{a(t)}{2\pi}} e^{-\frac{a(t)}{2}(x-m(t))^2}, \tag{3.6}$$

where

$$a(t) := \frac{a\sigma - \tan(2\sigma t)}{\sigma(1 + a\sigma\tan(2\sigma t))}, \quad m(t) := \frac{ma\sigma}{a\sigma\cos(2\sigma t) - \sin(2\sigma t)}. \tag{3.7}$$

Notice that $T < T^{\text{Heat}} = \frac{\pi}{4\sigma}$. Since $a(t) \searrow 0$ as $t \nearrow T$, it is easily seen that $u(t,x) \to 0$ uniformly in $x \in \mathbb{R}$. This extinction in finite time is actually true for *all* initial data, as stated in the following theorem.

Theorem 3.4 (Extinction in finite time)**.** *Let $u_0 \geq 0$, with $\int_{\mathbb{R}} u_0 = 1$. Let T be given by* (3.1)*.*

(*i*) *If $T = T^{\rm Heat}$, then in* (1.1)*, both u and $\overline{f}$ exist on $[0, T^{\rm Heat})$, in the sense that $u \in L^\infty_{\rm loc}((0,T^{\rm Heat})\times\mathbb{R})\cap C([0,T^{\rm Heat}); L^1(\mathbb{R}))$ and $\overline{f} \in C(0, T^{\rm Heat})$. The total mass is conserved before $T^{\rm Heat}$, $\int_{\mathbb{R}} u(t,x)dx = 1$ for all $0 \leq t < T^{\rm Heat}$. Moreover, extinction at time $T^{\rm Heat}$ occurs, that is*

$$u(t,x) = 0, \quad \forall t > T^{\rm Heat}, \ \forall x \in \mathbb{R}.$$

(*ii*) *If $0 < T < T^{\rm Heat}$, then extinction in finite time occurs:*

$$u(t,x) = 0, \quad \forall t > T, \ \forall x \in \mathbb{R}.$$

(*iii*) *If $T = 0$, then $u(t,x)$ is defined for no $t > 0$.*

3.2. **Proofs.**

Proof of Theorem 3.1. Like in the previous section, we can reduce (1.1) to the heat equation by combining two changes of unknown function. First, we have

$$u(t,x) = \frac{v(t,x)}{1 + \displaystyle\int_0^t \int_{\mathbb{R}} x^2 v(s,x)\, dxds}, \tag{3.8}$$

where $v(t,x)$ solves the Cauchy problem

$$\partial_t v = \sigma^2 \partial_{xx} v + x^2 v, \quad t > 0, \ x \in \mathbb{R}; \quad v(0,x) = u_0(x). \tag{3.9}$$

Notice that relation (3.8) is valid as long as $\overline{f}(t)$ remains finite. Next, we have

$$v(t,x) = \frac{1}{\sqrt{\cos(2\sigma t)}} e^{\frac{\tan(2\sigma t)}{2\sigma}x^2} w\left(\frac{\tan(2\sigma t)}{2\sigma}, \frac{x}{\sigma\cos(2\sigma t)}\right), \tag{3.10}$$

where $w(t,x)$ solves the heat equation

$$\partial_t w = \partial_{xx} w, \quad t > 0, \ x \in \mathbb{R}; \quad w(0,x) = u_0(\sigma x). \tag{3.11}$$

Notice that relation (3.10) is valid for $0 < t < T^{\rm Heat} = \frac{\pi}{4\sigma}$. Combining (3.8), (3.10) and the integral expression of w via the heat kernel, we end up with

$$u(t,x) = \frac{\sqrt{\frac{\sigma}{2\pi}} \frac{1}{\sqrt{\sin(2\sigma t)}} e^{\frac{\tan(2\sigma t)}{2\sigma}x^2} \displaystyle\int_{\mathbb{R}} e^{-\frac{\sigma}{2\tan(2\sigma t)}\left(\frac{x}{\sigma\cos(2\sigma t)} - y\right)^2} u_0(\sigma y)\, dy}{1 + I(t)}, \tag{3.12}$$

where

$$I(t) := \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} x^2 \sqrt{\frac{\sigma}{2\pi}} \frac{1}{\sqrt{\sin(2\sigma s)}} e^{\frac{\tan(2\sigma s)}{2\sigma}x^2} e^{-\frac{\sigma}{2\tan(2\sigma s)}\left(\frac{x}{\sigma\cos(2\sigma s)} - y\right)^2} u_0(\sigma y)\, dydxds.$$

Let us compute $I(t)$. Using Fubini's theorem, we first compute the integral with respect to x. Using elementary algebra (canonical form in particular), we get

$$\begin{aligned}
&\int_{\mathbb{R}} x^2 e^{\frac{\tan(2\sigma s)}{2\sigma}x^2} e^{-\frac{\sigma}{2\tan(2\sigma s)}\left(\frac{x}{\sigma\cos(2\sigma s)} - y\right)^2} dx \\
&= e^{\frac{\sigma\tan(2\sigma s)}{2}y^2} \int_{\mathbb{R}} x^2 e^{-\frac{1}{2\sigma\tan(2\sigma s)}\left(x - \frac{\sigma y}{\cos(2\sigma s)}\right)^2} dx \\
&= e^{\frac{\sigma\tan(2\sigma s)}{2}y^2} \sqrt{2\pi\sigma\tan(2\sigma s)} \left(\sigma\tan(2\sigma s) + \frac{\sigma^2 y^2}{(\cos(2\sigma s))^2}\right)
\end{aligned} \tag{3.13}$$

where we have used $\int_{\mathbb{R}} z^2 e^{-\frac{a}{2}(z-\theta)^2}dz = \frac{1}{a}\sqrt{\frac{2\pi}{a}} + \theta^2\sqrt{\frac{2\pi}{a}}$. Next, we pursue the computation of $I(t)$ and, integrating with respect to s, we find

$$\begin{aligned}
&\int_0^t \sqrt{\frac{\sigma}{2\pi}} \frac{1}{\sqrt{\sin(2\sigma s)}} e^{\frac{\sigma\tan(2\sigma s)}{2}y^2} \sqrt{2\pi\sigma\tan(2\sigma s)} \left(\sigma\tan(2\sigma s) + \frac{\sigma^2 y^2}{(\cos(2\sigma s))^2}\right) ds \\
&= \sigma \int_0^t e^{-\frac{1}{2}\ln(\cos(2\sigma s))} e^{\frac{\sigma\tan(2\sigma s)}{2}y^2} \left(\sigma\tan(2\sigma s) + \frac{\sigma^2 y^2}{(\cos(2\sigma s))^2}\right) ds \\
&= \sigma \int_0^t \frac{d}{ds}\left(e^{-\frac{1}{2}\ln(\cos(2\sigma s))} e^{\frac{\sigma\tan(2\sigma s)}{2}y^2}\right) ds \\
&= \sigma \left(\frac{1}{\sqrt{\cos(2\sigma t)}} e^{\frac{\sigma\tan(2\sigma t)}{2}y^2} - 1\right).
\end{aligned}$$

Finally, we integrate with respect to y and, using $\int_{\mathbb{R}} u_0 = 1$, get

$$\begin{aligned}
I(t) &= \int_{\mathbb{R}} \sigma \left(\frac{1}{\sqrt{\cos(2\sigma t)}} e^{\frac{\sigma\tan(2\sigma t)}{2}y^2} - 1\right) u_0(\sigma y)\, dy \\
&= \frac{1}{\sqrt{\cos(2\sigma t)}} \int_{\mathbb{R}} e^{\frac{\tan(2\sigma t)}{2\sigma}z^2} u_0(z)\, dz - 1.
\end{aligned}$$

Plugging this in the denominator of (3.12), and using the change of variable $z = \sigma y$ in the numerator of (3.12), we get (3.2), from which (3.3) easily follows. Using (3.3), Fubini theorem and the same computation as in (3.13), we obtain (3.4). Theorem 3.1 is proved. □

Proof of Proposition 3.3. The proof is rather similar to that of Proposition 2.2. It consists in plugging the Gaussian data (3.5) into formula (3.2) and using elementary algebra (canonical form). Details are omitted. □

Proof of Theorem 3.4. Let us assume $T = T^{\text{Heat}}$ and prove (i). Since

$$\int_{\mathbb{R}} e^{\frac{\tan(2\sigma t)}{2\sigma}y^2} u_0(y) dy < \infty \text{ for all } 0 < t < T^{\text{Heat}},$$

we have $\int_{\mathbb{R}} e^{\frac{\tan(2\sigma t)}{2\sigma}y^2} y^2 u_0(y)\, dy < \infty$ for all $0 < t < T^{\text{Heat}}$, and therefore both (3.4) and (3.3) are meaningful for all $0 < t < T^{\text{Heat}}$. It follows from (3.3) that

$$0 \le u(t,x) \le \frac{1}{\sqrt{2\pi\sigma\tan(2\sigma t)}},$$

and the right hand side goes to zero as $t \nearrow T^{\text{Heat}}$.

Let us assume $0 < T < T^{\text{Heat}}$ and prove (ii). It follows from (3.2) that

$$0 \le u(t,x) \le \frac{e^{\frac{\tan(2\sigma t)}{2\sigma}x^2}}{\sqrt{2\pi\sigma\tan(2\sigma t)} \int_{\mathbb{R}} e^{\frac{\tan(2\sigma t)}{2\sigma}y^2} u_0(y)\, dy},$$

and, the right hand side goes to zero as $t \nearrow T < T^{\text{Heat}}$.

Finally, assume $T = 0$ and prove (iii). Supposing by contradiction that there is a $\tau > 0$ such that $\overline{f}$ is finite on $[0,\tau]$, then (3.2) would hold true. On the other hand, the assumption $T = 0$, along with (3.2), would imply $u(t,x) = 0$ for all $t \in (0,\tau]$ and all $x \in \mathbb{R}$, while we know that so long as $\overline{f}$ is finite, we have $\int_{\mathbb{R}} u(t,x)dx = 1$, hence a contradiction. □

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